SCHOOL OF PUBLIC POLICY

CENTER FOR GLOBAL SUSTAINABILITY

Finance 201: The Unanswered Questions

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Introduction

- Finance 101 Review:
  - Two Core Elements: Budget, Finance
- SAV Financing Project Review
- Look Towards the Future
- The Third Element:
  - Economic Development

Finance 101 Review
Budget vs. Finance

- Completely connected, yet mutually exclusive

- Budget:
  - an amount of money *available* for spending based on how that money will be spent

  - An internal process

Finance 101 Review
Budget vs. Finance

- Completely connected, yet mutually exclusive

- Finance:
  - The allocation and distribution of fiscal resources with the goal of maximizing return on investment (ROI)

  - An external process

Submerged Aquatic Vegetation (SAV)
Finance Project Review

- Budget and Finance Workgroup Project

- Project Goal:
  - Develop a plan for funding and financing SAV program goals
- Project Strategy:
  - Couple finance experts with scientists and SAV experts
  - Three successively larger convenings culminating in a financing forum

Submerged Aquatic Vegetation (SAV)
Finance Project Review

- Budget and finance interaction
- Budget:
  - Funding the annual survey
- Finance:
  - Paying for survey results
  - SAV restoration and protection
  - Efficiency, monopolies, ecosystem service payments, revenue streams, institutional structures

Submerged Aquatic Vegetation (SAV)
Finance Project Review

- Lessons learned:
    - Survey: there is a market; finance it accordingly
    - Shortest distance between two points is an angle
        - Support water quality restoration
    - Find the market
    - It's not that complicated

MATHEMATICAL FINANCE CHEAT SHEET

Normal Random Variables

A random variable X is Normal $N(\mu,\sigma^2)$ (aka. *Gaussian*) under a measure $\mathbb{P}$ if and only if

$$\mathbb{E}_{\mathbb{P}}[e^{\theta X}] = e^{\theta\mu + \frac{1}{2}\theta^2\sigma^2}, \quad \text{for all real } \theta.$$

A standard normal $Z \sim N(0,1)$ under a measure $\mathbb{P}$ has density

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \qquad \mathbb{P}(Z \le z) = \Phi(z) = \int_{-\infty}^{z} \phi(x)\,dx.$$

Let $X = (X_1, X_2, \ldots, X_n)^T$ with $X_i \sim N(\mu_i, q_{ii})$ and $\mathrm{Cov}(X_i, X_j) = q_{ij}$ for $i,j = 1,\ldots,n$. We call $\mu = (\mu_1,\ldots,\mu_n)^T$ the *mean* and $Q = [q_{ij}]_{i,j=1}^{n}$ the *covariance matrix* of X. Assume $\det Q > 0$, then X has a *multivariate normal distribution* if it has the density

$$\phi(x) = \frac{1}{\sqrt{(2\pi)^n \det Q}} \exp\left(-\frac{1}{2}(x-\mu)^T Q^{-1}(x-\mu)\right), \quad x \in \mathbb{R}^n.$$

We write $X \sim N(\mu, Q)$ if this is the case. Alternatively, $X \sim N(\mu,Q)$ under $\mathbb{P}$ if and only if

$$\mathbb{E}_{\mathbb{P}}[e^{\theta^T X}] = \exp\left(\theta^T\mu + \frac{1}{2}\theta^T Q\theta\right), \quad \text{for all } \theta \in \mathbb{R}^n.$$

If $X \sim N(\mu, Q)$ and $c \in \mathbb{R}^n$ then $Y = c^T X \sim N(c^T\mu, c^T Q c)$. If $C \in \mathbb{R}^{m\times n}$ (i.e., $m \times n$ matrix) then $Y = CX \sim N(C\mu, CQC^T)$ and CQC^T is a $m \times m$ covariance matrix.

Gaussian Shifts

If $Z \sim N(0,1)$ under a measure $\mathbb{P}$, h is an integrable function, and c is a constant then

$$\mathbb{E}_{\mathbb{P}}[e^{cZ} h(Z)] = e^{c^2/2}\,\mathbb{E}_{\mathbb{P}}[h(Z+c)].$$

Let $X \sim N(0,Q)$, h be a integrable function of $x \in \mathbb{R}^n$, and $c \in \mathbb{R}^n$. Then

$$\mathbb{E}_{\mathbb{P}}[e^{c^T X} h(X)] = e^{\frac{1}{2}c^T Q c}\,\mathbb{E}_{\mathbb{P}}[h(X + Qc)].$$

Correlating Brownian Motions

Let $\{W(t)\}_{t\ge0}$ and $\{\tilde{W}(t)\}_{t\ge0}$ be independent Brownian motions. Given a correlation coefficient $\rho \in [-1,1]$, define

$$\hat{W}(t) := \rho W(t) + \sqrt{1-\rho^2}\,\tilde{W}(t),$$

then $\{\hat{W}(t)\}_{t\ge0}$ is a Brownian motion and $\mathbb{E}[W(t)\hat{W}(t)] = \rho t$.

Identifying Martingales

If $X_t = X(t)$ is a diffusion process satisfying

$$dX(t) = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW(t)$$

and $\mathbb{E}_{\mathbb{P}}[(\int_0^T \sigma(t,X_t)^2\,dt)^{1/2}] < \infty$ (or, $\sigma(t,x) \le c|x|$ as $|x| \to \infty$), then

X is a martingale $\iff$ X is driftless (i.e., $\mu(t) \equiv 0$ with $\mathbb{P}$-prob. 1).

Novikov's Condition

In the case $dX(t) = \sigma(t)X(t)\,dW(t)$ for some $\mathcal{F}$-previsible process $\{\sigma(t)\}_{t\ge0}$, then we have the simpler condition

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \sigma(t)^2\,dt\right)\right] < \infty \Rightarrow X \text{ is a martingale.}$$

Itô's Formula

For $X_t = X(t)$ given by $dX(t) = \mu(t)\,dt + \sigma(t)\,dW(t)$ and a function $g(t,x)$ that is twice differentiable in x and once in t. Then for $Y(t) = g(t, X_t)$, we have

$$dY(t) = \frac{\partial g}{\partial t}(t,X_t)\,dt + \frac{\partial g}{\partial x}(t,X_t)\,dX_t + \frac{1}{2}\sigma(t)^2\frac{\partial^2 g}{\partial x^2}(t,X_t)\,dt.$$

The Product Rule

Given $X(t)$ and $Y(t)$ adapted to the same Brownian motion $\{W(t)\}_{t\ge0}$.

$$dX(t) = \mu(t)\,dt + \sigma(t)\,dW(t), \quad dY(t) = \nu(t)\,dt + \rho(t)\,dW(t).$$

Then $d(X(t)Y(t)) = X(t)\,dY(t) + Y(t)\,dX(t) + \underbrace{d[X,Y](t)}_{\sigma(t)\rho(t)\,dt}$.

In the other case, if $X(t)$ and $Y(t)$ are adapted to two different and independent Brownian motions $\{W(t)\}_{t\ge0}$ and $\{\tilde{W}(t)\}_{t\ge0}$.

$$dX(t) = \mu(t)\,dt + \sigma(t)\,dW(t), \quad dY(t) = \nu(t)\,dt + \rho(t)\,d\tilde{W}(t).$$

Then $d(X(t)Y(t)) = X(t)\,dY(t) + Y(t)\,dX(t)$ as $d[X,Y](t) = 0$.

Radon-Nikodým Derivative

Given $\mathbb{P}$ and $\mathbb{Q}$ equivalent measures and a time horizon T, we can define a random variable $\frac{d\mathbb{Q}}{d\mathbb{P}}$ defined on $\mathbb{P}$-possible paths, taking positive real values, such that

- $\mathbb{E}_{\mathbb{Q}}[X_T] = \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}X_T\right]$, for all claims X_T knowable by time T,
- $\mathbb{E}_{\mathbb{Q}}[X_t|\mathcal{F}_s] = \zeta_s^{-1}\mathbb{E}_{\mathbb{P}}[\zeta_t X_t|\mathcal{F}_s]$, for $s \le t \le T$,

where ζ_t is the process $\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t]$.

Cameron-Martin-Girsanov Theorem

If $\{W(t)\}_{t\ge0}$ is a $\mathbb{P}$-Brownian motion and $\{\gamma(t)\}_{t\ge0}$ is an $\mathcal{F}$-previsible process satisfying the boundedness condition $\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \gamma(t)^2\,dt\right)\right] < \infty$, then there exists a measure $\mathbb{Q}$ such that

- $\mathbb{Q}$ is equivalent to $\mathbb{P}$,
- $\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \gamma(t)\,dW(t) - \frac{1}{2}\int_0^T \gamma(t)^2\,dt\right)$,
- $\tilde{W}(t) = W(t) + \int_0^t \gamma(s)\,ds$ is a $\mathbb{Q}$-Brownian motion.

In other words, $W(t)$ is a drifting $\mathbb{Q}$-Brownian motion with drift $-\gamma(t)$ at time t.

Cameron-Martin-Girsanov Converse

If $\{W(t)\}_{t\ge0}$ is a $\mathbb{P}$-Brownian motion, and $\mathbb{Q}$ is a measure equivalent to $\mathbb{P}$, then there exists a $\mathcal{F}$-previsible process $\{\gamma(t)\}_{t\ge0}$ such that

$$\tilde{W}(t) = W(t) + \int_0^t \gamma(s)\,ds$$

is a $\mathbb{Q}$-Brownian motion. That is, $W(t)$ plus drift $\gamma(t)$ is a $\mathbb{Q}$-Brownian motion. Additionally,

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \gamma(t)\,dW(t) - \frac{1}{2}\int_0^T \gamma(t)^2\,dt\right).$$

Martingale Representation Theorem

Suppose $\{M(t)\}_{t\ge0}$ is a $\mathbb{Q}$-martingale process whose volatility $\sqrt{\mathbb{E}_{\mathbb{Q}}[M(t)^2]} = \sigma(t)$ satisfies $\sigma(t) \neq 0$ for all t (with $\mathbb{Q}$-probability one). Then if $\{N(t)\}_{t\ge0}$ is any other $\mathbb{Q}$-martingale, there exists an $\mathcal{F}$-previsible process $\{\phi(t)\}_{t\ge0}$ such that $\int_0^T \phi(t)^2\sigma(t)^2\,dt < \infty$ (with $\mathbb{Q}$-prob. one), and N can be written as

$$N(t) = N(0) + \int_0^t \phi(s)\,dM(s),$$

or in differential form, $dN(t) = \phi(t)\,dM(t)$. Further, ϕ is (essentially) unique.

Multidimensional Diffusions, Quadratic Covariation, and Itô Formula

If $X := (X_1, X_2, \ldots, X_n)^T$ is a n-dimensional diffusion process with form

$$X(t) = X(0) + \int_0^t \mu(s)\,ds + \int_0^t \Sigma(s)\,dW(s),$$

where $\Sigma(s) \in \mathbb{R}^{n\times m}$ and W is a m-dimensional Brownian motion. The *quadratic covariation* of the components X_i and X_j is

$$[X_i, X_j](t) = \int_0^t \Sigma_i(s)^T\Sigma_j(s)\,ds,$$

or in differential form $d[X_i, X_j](t) = \Sigma_i(t)^T\Sigma_j(t)\,dt$, where $\Sigma_i(t)$ is the i^{th} column of $\Sigma(t)$. The quadratic variation of $X_i(t)$ is $[X_i](t) = \int_0^t \Sigma_i(s)^T\Sigma_i(s)\,ds$.
The *multi-dimensional Itô formula* for $Y(t) = f(t, X_1(t), \ldots, X_n(t))$ is

$$dY(t) = \frac{\partial f}{\partial t}(t, X_1(t), \ldots, X_n(t))\,dt + \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(t, X_1(t), \ldots, X_n(t))\,dX_i(t)$$
$$+ \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_1(t), \ldots, X_n(t))\,d[X_i, X_j](t).$$

The *(vector-valued) multi-dimensional Itô formula* for

$$Y(t) = f(t, X(t)) = (f_1(t, X(t)), \ldots, f_m(t, X(t)))^T$$

where $f_k(t,X) = f_k(t, X_1, \ldots, X_n)$ and $Y(t) = (Y_1(t), Y_2(t), \ldots, Y_m(t))^T$ is given component-wise (for $k = 1,\ldots,m$) as

$$dY_k(t) = \frac{\partial f_k(t,X(t))}{\partial t}\,dt + \sum_{i=1}^{n}\frac{\partial f_k(t,X(t))}{\partial x_i}\,dX_i(t)$$
$$+ \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f_k(t,X(t))}{\partial x_i\partial x_j}\,d[X_i, X_j](t).$$

Stochastic Exponential

The *stochastic exponential* of X is $\mathcal{E}_t(X) = \exp(X(t) - \frac{1}{2}[X](t))$. It satisfies

$$\mathcal{E}(0) = 1, \quad \mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X+Y)e^{[X,Y]}, \quad \mathcal{E}(X)^{-1} = \mathcal{E}(-X)e^{[X,X]}.$$

The process $Z = \mathcal{E}(X)$ is a positive process and solves the SDE

$$dZ = Z\,dX, \quad Z(0) = e^{X(0)}.$$

Solving Linear ODEs

The linear *ordinary differential equation*

$$\frac{dx(t)}{dt} = m(t) + \mu(t)x(t), \quad x(a) = \zeta,$$

for $a \le t \le b$ has solution given by

$$x(t) = \zeta e_t + \int_a^t e_t e_u^{-1} m(u)\,du, \qquad e_t = \exp\left(\int_a^t \mu(u)\,du\right),$$
$$= \zeta\exp\left(\int_a^t \mu(u)\,du\right) + \int_a^t m(u)\exp\left(\int_u^t \mu(r)\,dr\right)du.$$

Solving Linear SDEs

The linear *stochastic differential equation*

$$dX(t) = [m(t) + \mu(t)X(t)]\,dt + [q(t) + \sigma(t)X(t)]\,dW(t), \quad X(a) = \zeta,$$

for $a \le t \le b$ has solution given by

$$X(t) = \zeta\mathcal{E}_t + \int_a^t \mathcal{E}_t\mathcal{E}_u^{-1}(m(u) - q(u)\sigma(u))\,du + \int_a^t \mathcal{E}_t\mathcal{E}_u^{-1}q(u)\,dW(u),$$

where $\mathcal{E}_t := \mathcal{E}_t(X)$ and $X(t) = \int_a^t \mu(u)\,du + \int_a^t \sigma(u)\,dW(u)$. In other words,

$$\mathcal{E}_t = \exp\left(\int_a^t \mu(u)\,du + \int_a^t \sigma(u)\,dW(u) - \frac{1}{2}\int_a^t \sigma(u)^2\,du\right).$$

Fundamental Theorem of Asset Pricing

Let X be some $\mathcal{F}_T$-measurable claim, payable at time T. The arbitrage-free price V of X at time t is

$$V(t) = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\,ds\right)X\,\middle|\,\mathcal{F}_t\right],$$

where $\mathbb{Q}$ is the risk-neutral measure.

Market Price Of Risk

Let $X_t = X(t)$ be the price of a non-tradable asset with dynamics $dX(t) = \mu(t)\,dt + \sigma(t)\,dW(t)$ where $\{\sigma(t)\}_{t\ge0}$ and $\{\mu(t)\}_{t\ge0}$ are previsible processes and $\{W(t)\}_{t\ge0}$ is a $\mathbb{P}$-Brownian motion. Let $Y(t) := f(X_t)$ be the price of a tradable asset where $f: \mathbb{R} \to \mathbb{R}$ is a derivable function. Then the *market price of risk* is

$$\gamma(t) := \frac{\mu_t f'(X_t) + \frac{1}{2}\sigma_t^2 f''(X_t) - r f(X_t)}{\sigma_t f'(X_t)},$$

and the behaviour of X_t under the risk-neutral measure $\mathbb{Q}$ is given by

$$dX(t) = \sigma(t)\,d\tilde{W}(t) + \frac{r f(X_t) - \frac{1}{2}\sigma_t^2 f''(X_t)}{f'(X_t)}\,dt.$$

Black Model

Consider a European option with strike price K on a asset with value V_T at maturity time T. Let F_0 be the forward price of V_T, F_0 the current forward price. If $\log V_T \sim N(F_0, \sigma^2 T)$ then the Call and Put prices are given by

$$\mathcal{C} = P(0,T)[F_0\Phi(d_1) - K\Phi(d_2)], \quad \mathcal{P} = P(0,T)[K\Phi(-d_2) - F_0\Phi(-d_1)],$$

where $d_1 = \frac{\log(\mathbb{E}_{\mathbb{Q}}(V_T)/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

Forward Rates, Short Rates, Yields, and Bond Prices

The *forward rate* at time t that applies between times T and S is defined as

$$F(t,T,S) = \frac{1}{S-T}\log\frac{P(t,T)}{P(t,S)}.$$

The *instantaneous forward rate* at time t is $f(t,T) = \lim_{S\to T} F(t,T,S)$. The *instantaneous risk-free rate* or *short rate* is $r(t) = \lim_{T\to t} f(t,T)$. The *cash account* is given by

$$B(t) = \exp\left(\int_0^t r(s)\,ds\right),$$

and satisfies $dB(t) = r(t)B(t)\,dt$ with $B(0) = 1$. The instantaneous forward rates and the yield can be written in terms of the bond prices as

$$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T), \quad R(t,T) = -\frac{\log P(t,T)}{T-t}.$$

Conversely,

$$P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right) \quad \text{and} \quad P(t,T) = \exp(-(T-t)R(t,T)).$$

Affine Jump Diffusion (AJD) Models

The state vector X_t follows a Markov process solving the SDE

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t + dZ_t$$

where W is an adapted Brownian, and Z is a pure jump process with intensity λ. The moment generating function of the jump sizes is $\theta(c) = \mathbb{E}_{\mathbb{Q}}[\exp(cJ)]$. Impose an affine structure on μ, $\sigma\sigma^T$, λ and the discount rate R, possibly time dependent:

$$\mu(x) = K_0 + K_1 x \quad (\sigma(x)\sigma(x)^T)_{ij} = (H_0)_{ij} + (H_1)_{ij}\,x \quad \lambda(x) = L_0 + L_1 x \quad R(x) = R_0 + R_1 x$$

Given X_0, the risk-neutral coefficients (K, H, L, θ, R) completely determine the discounted risk-neutral distribution of X. Introduce the transform function

$$\psi(u, X_0, T) = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_0^T R(X_s)\,ds\right)e^{u^T X_T}\,\middle|\,\mathcal{F}_0\right] = e^{\alpha(0,u) + \beta(0,u)^T X_0}$$

where α and β solve the Ricatti ODEs subject to $\alpha(T,u) = 0$, $\beta(T,u) = u$

$$-\dot{\beta}(t,u) = K_1^T\beta(t,u) + \frac{1}{2}\beta(t,u)^T H_1\beta(t,u) + L_1(\theta(\beta(t,u)) - 1) - R_1$$
$$-\dot{\alpha}(t,u) = K_0^T\beta(t,u) + \frac{1}{2}\beta(t,u)^T H_0\beta(t,u) + L_0(\theta(\beta(t,u)) - 1) - R_0$$

AJD bond pricing

In ψ, set $L_1 = R_0 = u = 0$, $R_1 = 1$ to obtain the zero coupon bond with maturity $T - t$ via the Ricatti ODEs:

Short rate model	K_0	K_1	H_0	H_1	P, MR?
Merton	μ		σ^2		N, N
Dothan		μ		σ^2	Y, N
Vasicek	$\alpha\mu$	$-\alpha$	σ^2		N, Y
CIR	$\alpha\mu$	$-\alpha$		σ^2	Y, Y
Pearson-Sun	$\alpha\mu$	$-\alpha$	$-\sigma^2\beta$	σ^2	Y, Y
Ho & Lee	$\theta(t)$		σ^2		N, N
Hull & White	$\alpha\mu(t)$	$-\alpha$	σ^2		N, Y
Extended Vasicek	$\alpha(t)\mu(t)$	$-\alpha(t)$	$\sigma(t)^2$		N, Y
Black-Karasinski†	$\alpha(t)\mu(t)$	$-\alpha(t)$	$\sigma(t)^2$		Y, Y

P means the process stays positive, MR means r_t is mean-reverting. Closed form solutions for bond prices and European options exist for all models except for †, which describes the evolution of $d\log(r_t)$ instead of dr_t.

AJD option pricing

Define the Fourier transform in terms of the modified expectation

$$G(a,b,y) = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_0^T R(X_s)\,ds\right)e^{a^T X_T}\mathbb{1}_{b\cdot X_T \le y}\right]$$
$$= \frac{\psi(a, X_0, T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{\Im[\psi(a + ivb, X_0, T)e^{-ivy}]}{v}\,dv$$

The ith entry in X is the log asset price and $k = \log(K)$, the log strike. d is a vector whose ith element is 1, else zero. The corresponding call option price is

$$C = G(d, -d, -k) - KG(0, -d, -k)$$

The Heath-Jarrow-Morton Framework

Given a initial forward curve $T \mapsto f(0,T)$ then, for every maturity T and under the real-world probability measure $\mathbb{P}$, the forward rate process $t \mapsto f(t,T)$ follows

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)\,ds + \int_0^t \sigma(s,T)^T\,dW(s), \quad t \le T,$$

where $\alpha(t,T) \in \mathbb{R}$ and $\sigma(t,T) := (\sigma_1(t,T), \ldots, \sigma_d(t,T))$ satisfy the technical conditions: (1) α and σ are previsible and adapted to $\mathcal{F}_t$; (2) $\int_0^T\int_0^T |\alpha(t,s)|\,ds\,dt < \infty$ for all T; (3) $\sup_{s,t\le T}\|\sigma(s,t)\| < \infty$ for all T. The short rate process is given by

$$r(t) = f(t,t) = f(0,t) + \int_0^t \alpha(s,t)\,ds + \int_0^t \sigma(s,t)\,dW(s),$$

so the cash account and zero coupon T-bond prices are well-defined and obtained through

$$B(t) = \exp\left(\int_0^t r(s)\,ds\right), \quad P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right).$$

The discounted asset price $Z(t,T) = P(t,T)/B(t)$ satisfies

$$dZ(t,T) = Z(t,T)\Big[\underbrace{\left(\frac{1}{2}S^2(t,T) - \int_t^T \alpha(t,u)\,du\right)}_{b(t,T)}dt + S(t,T)^T\,dW(t)\Big],$$

where $S(t,T) := -\int_t^T \sigma(t,u)\,du$. The *HJM drift condition* states that

$\mathbb{Q}$ is EMM (i.e., no arbitrage for bonds) $\iff$ $b(t,T) = -S(t,T)\gamma(t)^T$,

where $\tilde{W}(t) := W(t) - \int_0^t \gamma(s)\,ds$ is a $\mathbb{Q}$-Brownian motion. If this holds, then under $\mathbb{Q}$, the forward rate process follows

$$f(t,T) = f(0,T) + \int_0^t \underbrace{\left(\sigma(s,T)\int_s^T \sigma(s,u)^T\,du\right)}_{\text{HJM drift}}ds + \int_0^t \sigma(s,T)\,d\tilde{W}(s),$$

and the discounted asset $Z(t,T)$ satisfies $dZ(t,T) = Z(0,T)\mathcal{E}_t(X)$ with

$$X(t) = \int_0^t S(s,T)\,d\tilde{W}(s).$$

The LIBOR Market Model

For a tenor $\delta > 0$, the *LIBOR rate* $L(T,T,T+\delta)$ is the rate such that an investment of 1 at time T will grow to $1 + \delta L(T,T,T+\delta)$ at time $T+\delta$. The *forward LIBOR rate* (i.e., a contract made at time t under which we pay 1 at time T and receive back $1 + \delta L(t,T,T+\delta)$ at time $T+\delta$) is defined as

$$L(t,T) := L(t,T,T+\delta) = \frac{1}{\delta}\left(\frac{P(t,T)}{P(t,T+\delta)} - 1\right),$$

and satisfies $L(T,T) = L(T,T,T+\delta)$.
Under the real-world probability measure $\mathbb{P}$, The LMM assumes that each LIBOR process $\{L(t,T_m)\}_{0\le t\le T_m}$ satisfies

$$dL(t,T_m) = L(t,T_m)\left[\mu(t, L(t,T_m))\,dt + \lambda_m(t, L(t,T_m))^T\,dW(t)\right],$$

where $W = (W^1, \ldots, W^d)$ is a d-dimensional Brownian motion with instantaneous correlations

$$d[W^i, W^j](t) = \rho_{i,j}(t)\,dt, \quad i,j = 1,2,\ldots,d.$$

The function $\lambda(t,L): [0,T_j] \times \mathbb{R} \to \mathbb{R}^{d\times d}$ is the volatility, and $\mu(t): [0,T_j] \to \mathbb{R}$ is the drift.
Let $0 \le m, n \le N-1$. Then the dynamics of $L(t,T_m)$ under the forward measure $\mathbb{P}_{T_{n+1}}$ is for $m < n$ given by

$$dL(t,T_m) = L(t,T_m)\left[-\lambda(t,T_m)\sum_{j=m+1}^{n}\sigma_{T_j,T_{j+1}}(t)^T\,dt + \lambda(t,T_m)\,dW^{n+1}(t)\right]$$

For $m = n$,

$$dL(t,T_m) = L(t,T_m)\lambda(t,T_m)\,dW_t^{m+1}$$

and for $m > n$ we have

$$dL(t,T_m) = L(t,T_m)\left[\lambda(t,T_m)\sum_{j=n+1}^{m}\sigma_{T_j,T_{j+1}}(t)^T\,dt + \lambda(t,T_m)\,dW_t^{m+1}\right]$$

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Finance Project Review

- Lessons learned:
  - Survey: there is a market; finance it accordingly
  - Shortest distance between two points is an angle
    - Support water quality restoration
  - Find the market
  - It's not that complicated
  - Internally: develop entrepreneurs
  - Externally: focus ***everything*** on investment

Finance System

The allocation and distribution of fiscal resources with the goal of ***maximizing return on investment*** (ROI)

Finance System

ROI: restored and protected Chesapeake Bay

- Scale
- Efficiency
- Innovation
- Ingenuity

Financing System

Scale

- Inter-jurisdictional financing
- Multiple revenue streams, public and private

Financing Systems

Efficiency

- Institutions
- Expertise across disciplines
- Metrics
- Flexibility

Financing Systems

Innovation

- Focus on outcomes, not outputs
- Link investment to science: metrics
- Financial reward

Financing Systems

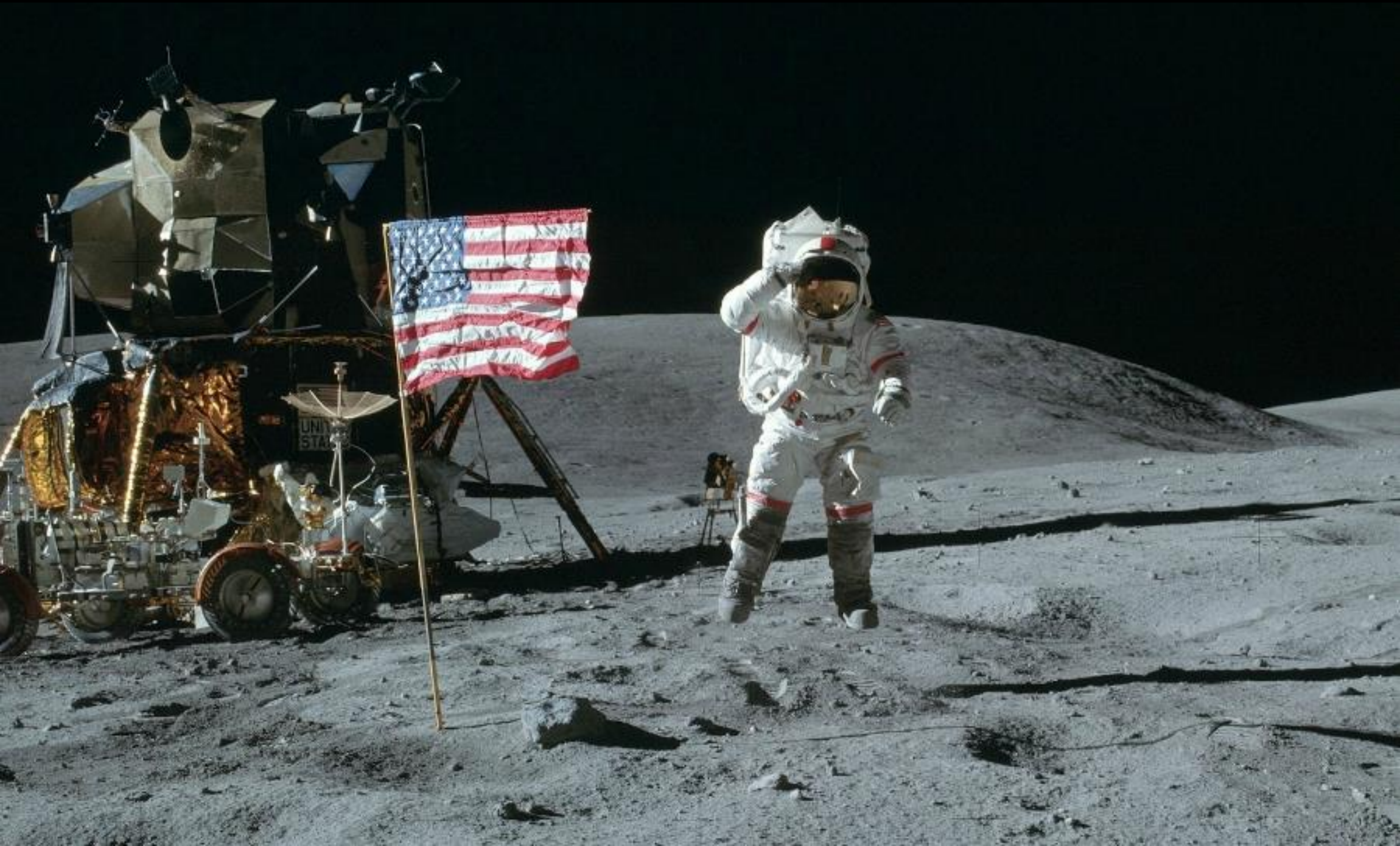
Ingenuity

- Sense of urgency
- Truth
- Myopic focus on the outcome and investment
- We must be bold

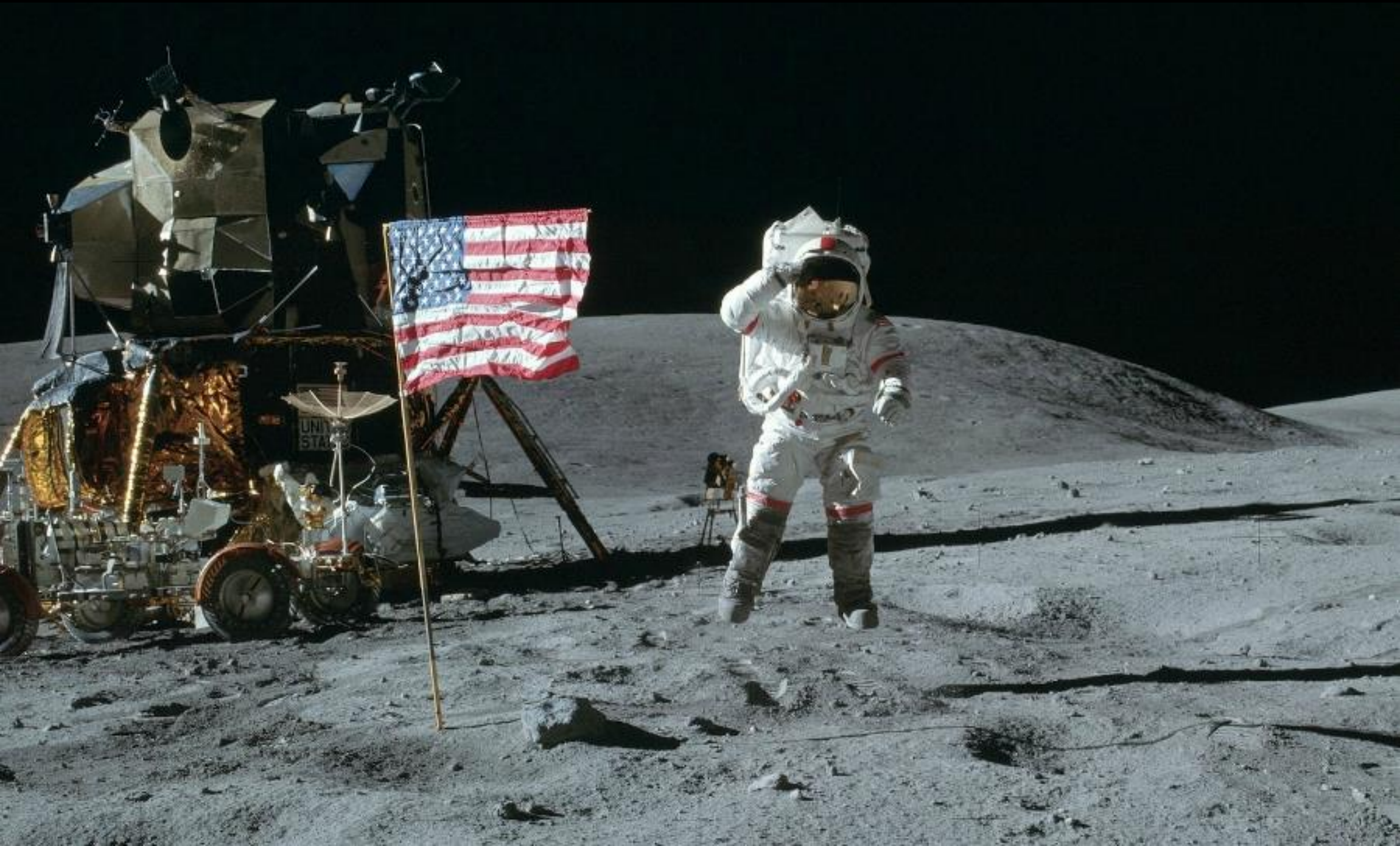
A Look Towards the Future

- Budgeting
- Finance
- Economy

Apollo Program: 400K+ employed



Apollo Program: 6,300 Inventions



Apollo Program:

- Impossible vision to achieve
- It was a true economic bubble
- Set the foundation for the future

Conclusion

SCHOOL OF PUBLIC POLICY

CENTER FOR GLOBAL SUSTAINABILITY

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